Next goal: CLT

Strong LLN $\frac{S_n - \mathbb{E}S_n}{n} \xrightarrow{a.s.} 0$ so $S_n - \mathbb{E}S_n$ grows slower than n. How fast does it grow? Turns out $\sqrt{n}$, so get a non-trivial limit when consider $\frac{S_n - \mathbb{E}S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{}$. In fact not deterministic anymore.

So S_n is of order $n\mathbb{E}X_1$ w/ fluctuations of order $\sqrt{n}$.

Thm (CLT) Let $X_1, X_2, \ldots$ be iid real-valued RVs w/ finite 2nd moment. Let $S_n = X_1 + \cdots + X_n$. Then

$$\frac{S_n - n\mathbb{E}X_1}{\sqrt{n}} \Rightarrow \mathcal{N}(0, \mathrm{Var}(X_1))$$

Need to study c/ce in distr, related to weak c/ce

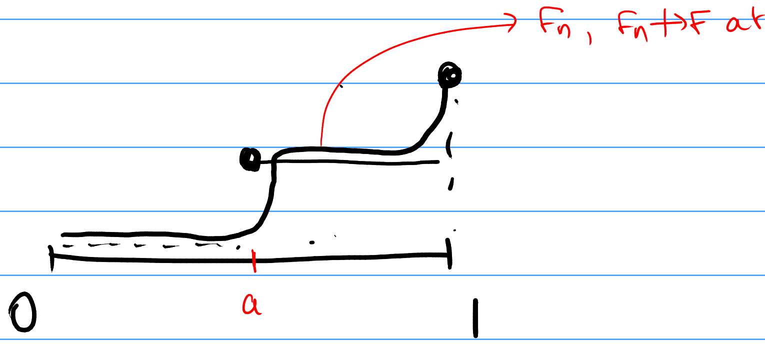
Weak c/ce

Defn: Given metric sp $\mathcal{X}$, let $C_c(\mathcal{X})$ be sp. of compactly supported cts fns, $C_b(\mathcal{X})$ sp. of bdd cts fns.

Defn: Let $\mathcal{X}$ be a top'l sp. & $\mu, \mu_1, \mu_2, \ldots$ be prob m'res on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Say μ_n c/v to μ weakly ($\mu_n \Rightarrow \mu$) if

$$\lim_{n\to\infty} \int f \, d\mu_n = \int f \, d\mu \quad \forall f \in C_b(\mathcal{X}).$$

Note: we have defined convergence in distribution before. $X_n \xrightarrow{d} X$ if $F_{X_n}(t) \longrightarrow F_X(t)$ $\forall t$ that's a pt of continuity of F_X.



What we have defined here is weak convergence of measures.

convergence in dist $\Leftrightarrow$ weak convergence.

One would imagine that one can take the standard route: intervals, simple fns, bounded functions.

A note on weak convergence of MEASURES.

There is also weak convergence in Banach spaces!

Weak convergence is closely related to something called vague convergence. In fact vague convergence + tightness $\Leftrightarrow$ weak convergence.

Vague convergence has the "advantage" of being a weak-$*$ convergence which fits in nicely into classical theory of convergence on vector spaces (of random variables and measures)

Let X be locally compact Hausdorff space. Compact Hausdorff spaces are normal and hence metrizable by Urysohn. So might as well assume metric space. Typically we work on complete, separable metric spaces, or Polish spaces.

Let $M(X)$ be the space of Radon measures on X
$C_0(X)$ be the Banach space of continuous functions on X vanishing at infinity. $\|f\|_\infty$ Norm.

By Riesz-Markov-Kakutani theorem, $M(X)$ is isometrically isomorphic to $C_0(X)^*$

$$\underline{I} : M(X) \longrightarrow C_0(X)^*$$

Recall $C_0(X)^*$ is the space of bounded linear functionals on $C_0(X)$. So the isometry $\underline{I}$ is as follows

$$\underline{I}(\mu) \text{ is the functional } \quad \underline{I}(\mu)(f) = \int_X f \, d\mu$$

The vague topology is the weak-$*$ topology on $C_0(X)^*$; i.e.,
$\mu_n \longrightarrow \mu$ (w-$*$) if

$$\int f \, d\mu_n \longrightarrow \int f \, d\mu \quad \forall \; f \in C_0(X)$$

It is not uncommon to define this convergence in terms of $f \in C_c(X)$ (compact support functions)

Thm: Let X_n rvs, μ_n distribution of X_n. $X_n \xrightarrow{d} X$ iff $\mu_n \xrightarrow{w} \mu$.

Pf: F is right continuous. Let $[a,b)$ be an interval. Let $(a,b]$ be points of continuity of F. Then

$$\mu_n(a,b] = F_n(b) - F_n(a) \xrightarrow{n\to\infty} \mu(a,b]$$

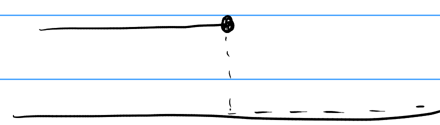
Let f be bounded continuous, and let φ_n (step fn) approximate f on a compact interval $[-K,K]$.

$$\int_{[-K,K]} \varphi \, d\mu_n \longrightarrow \int_{[-K,K]} \varphi \, d\mu \quad \Rightarrow \quad \int f \, d\mu_n \longrightarrow \int f \, d\mu.$$

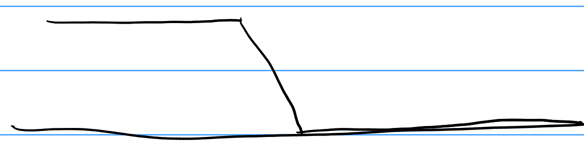
Uniform continuity and boundedness of f

The other way around is easy too. Approximate

Approximate $1_{(-\infty,x]}$ =



with



That's it. If you want to see a full proof do Sevah's / Khoshnevisan below.

Def: RVs X, X_n have distr μ, μ_n & $\mu_n \Rightarrow \mu$, say X_n cv weakly to X & write $X_n \Rightarrow X$. Eq. to

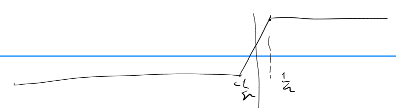
$$\mathbb{E} f(X_n) \to \mathbb{E} f(X) \quad \forall \text{ bdd cts } f.$$

Thm: let $\mu, \mu_1, \mu_2, \ldots$ be prob mbles on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ w/ distr fns $F, F_1, F_2, \ldots$ resp. Then $\mu_n \Rightarrow \mu$ iff $F_n(x) \to F(x)$ $\forall x$ where F is cts

(i.e. weak cv'ce is eq. to cv'ce in distr.)

Ex: Unif on $[-\frac{1}{n}, \frac{1}{n}] \xRightarrow{\text{weak}}$ pt mass at 0.

but



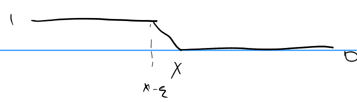
$\forall x \neq 0$ so do need to exclude pts of discont.

$F_n(0) = 1/2$

Pf of thm: let X_n, X be RVs w/ distr μ_n, μ resp.

$(\Rightarrow)$ Suppose $X_n \Rightarrow X$. Let x be such that F cts at x. Let $\varepsilon > 0$

let f be



the function drawn in the picture.

$$f(y) \le \mathbb{1}_{(-\infty, x]}(y) \le f(y-\varepsilon) \quad \forall y \in \mathbb{R} \qquad - (\star 1)$$

$\Rightarrow$

$$\mathbb{E} f(X_n) \le F_n(x) \le \mathbb{E} f(X_n - \varepsilon)$$

$\downarrow$

$$\mathbb{E} f(X) \le \liminf F_n(x) \le \limsup F_n(x) \le \mathbb{E} f(X - \varepsilon)$$

Need to connect $f(X)$ to F. $(\star 1)$ also implies

$$\mathbb{1}_{(-\infty, x]}(y) \le f(y-\varepsilon) \Leftrightarrow \mathbb{1}_{(-\infty, x-\varepsilon]}(y) \le f(y) \qquad f(y-\varepsilon) \le \mathbb{1}_{(-\infty, x+\varepsilon]}(y) \quad (\gamma)$$

plug in x and take the expectation $\xrightarrow{\text{so}}$

$$F(x-\varepsilon) \le \mathbb{E} f(X) \qquad \mathbb{E} f(X-\varepsilon) \le F(x+\varepsilon)$$

so

$$F(x-\varepsilon) \le \liminf F_n(x) \le \limsup F_n(x) \le F(x+\varepsilon)$$

$\varepsilon \downarrow 0$ done (using x is a pt of continuity)

$(\Leftarrow)$ Suppose $F_n(x) \to F(x)$ $\forall x$ where F is cts.

Let f be bdd, cts. $\mathbb{E} f(X_n) \to \mathbb{E} f(X)$

$\downarrow$

need to show

Let $\delta > 0$. $\exists\, a, b \in \mathbb{R}$ s.t. $F(a) < \delta$ & $F(b) > 1-\delta$.

f cts on $[a,b] \Rightarrow$ unif. cts. $\Rightarrow \exists\, \alpha > 0$ s.t. $|x-y| < \alpha$, $x,y \in [a,b]$

implies $|f(x) - f(y)| < \delta$.

Can you show this?

F has at most ctbly many discontinuities so can divide $[a,b]$ into pieces $<\alpha$ avoiding discon. of F. Say $a = x_0 < x_1 < \dots < x_m = b$ s.t. F cts at $x_i\ \forall i$ & $|x_i - x_{i-1}| < \alpha$.

Approximate $\mathbb{E}f$ using the pts $\{x_i\}$ only.

fixed pts

Then
$$\left| \mathbb{E}[f(X_n) 1_{(a,b]}(X_n)] - \sum_{i=1}^m f(x_i)[F_n(x_i) - F_n(x_{i-1})] \right|$$

r.v.

$$= \left| \sum_{i=1}^m \left(\mathbb{E}[f(X_n) 1_{(x_{i-1}, x_i]}(X_n)] - f(x_i)[F_n(x_i) - F_n(x_{i-1})] \right) \right|$$

uses uniform continuity of f

$$= \left| \sum_{i=1}^m \mathbb{E}\left([f(X_n) - f(x_i)] 1_{(x_{i-1}, x_i]}(X_n) \right) \right| \le \sum_{i=1}^m \delta P(X_n \in (x_{i-1}, x_i]) \le \delta.$$

Similarly
$$\left| \mathbb{E}[f(X) 1_{(a,b]}(X)] - \sum_{i=1}^m f(x_i)[F(x_i) - F(x_{i-1})] \right| \le \delta$$

m - fixed so F cts at $x_i\ \forall i$ so $F_n(x_i) \to F(x_i)$ so
$$\lim_{n\to\infty} \sum_{i=1}^m f(x_i)[F_n(x_i) - F_n(x_{i-1})] = \sum_{i=1}^m f(x_i)[F(x_i) - F(x_{i-1})].$$

Thus
$$\limsup_{n\to\infty} \left| \mathbb{E}[f(X_n) 1_{(a,b]}(X_n)] - \mathbb{E}[f(X) 1_{(a,b]}(X)] \right| \le 2\delta.$$

If f bdd by K, get
$$\mathbb{E}[f(X_n) 1_{(a,b]^c}(X_n)] \le K \cdot 2\delta \quad \text{\& same w/o } n.$$

Thus
$$\limsup_{n\to\infty} \left| \mathbb{E}[f(X_n)] - \mathbb{E}[f(X)] \right| \le 2\delta + 4K\delta$$
$\delta \to 0$. $\triangle$

$\mu_n \Rightarrow \mu$ means $\mathbb{E} f(X_n) \to \mathbb{E} f(X)$ $\forall f \in C_b(X)$.

We have $C_c(X) \subseteq C_b(X)$. Turns out $f \in C_c(X)$ is enough.

See note about vague convergence.

Thm: $\mu, \mu_1, \mu_2, \ldots$ prob measures on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. Then $\mu_n \Rightarrow \mu$ iff

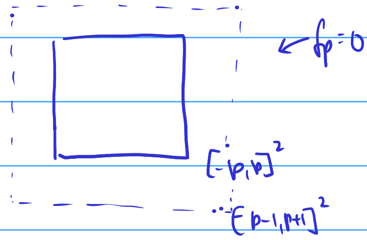
$$\lim_{n\to\infty} \int f\, d\mu_n = \int f\, d\mu \quad \forall\, f \in C_c(\mathbb{R}^k).$$

Pf: Let $f \in C_b(\mathbb{R}^k)$. NTS $\int f\, d\mu_n \to \int f\, d\mu$.

WLOG $f \geq 0$.

Want to approximate f by compactly supported cts fns.

For $p \in \mathbb{N}$ let f_p be such that



$$f_p(x) = f(x) \quad \forall\, x \in [-p,p]^k$$
$$f_p(x) = 0 \quad \forall\, x \in ([-p-1,p+1]^k)^c$$
$$\& \ 0 \leq f_p(x) \leq f(x) \quad \text{o/w}.$$

Have $f_p(x) \uparrow f(x)$ as $p \to \infty$. $\Rightarrow$

$$\liminf_{n\to\infty} \int f\, d\mu_n \geq \lim_{n\to\infty} \int f_p\, d\mu_n = \int f_p\, d\mu \quad \text{since } f_p \in C_c(\mathbb{R}^k).$$

$p \uparrow \infty$ gives $\liminf_{n\to\infty} \int f\, d\mu_n \geq \int f\, d\mu$ To show $\int f\, d\mu \geq \overline{\lim} \int f\, d\mu_n$

On the other hand

$$\int f\, d\mu_n = \int_{[-p,p]^k} f\, d\mu_n + \int_{\mathbb{R}^k \setminus [-p,p]^k} f\, d\mu_n$$

$$\leq \underbrace{\int_{[-p,p]^k} f_p\, d\mu_n}_{\leq \int f_p\, d\mu_n} + \underbrace{\sup_{z \in \mathbb{R}^k} |f(z)|}_{K} \left(1 - \mu_n([-p,p]^k)\right)$$

Instead lets reuse Davar's argument: Approximate $1_{[-p,p]^n}$ by continuous functions from below. Call these $\{\psi_m\}$, $\psi_m \to 1_{[-p,p]^n}$

$$\int 1_{[-p,p]^n}\, d\mu_n \geq \int \psi_m\, d\mu_n \Rightarrow$$

$$\varliminf_{n\to\infty} \int 1_{[-p,p]^n}\, d\mu_n \geq \varliminf_{n\to\infty} \int \psi_m \, d\mu_n = \int \psi_m \, d\mu.$$

Then take a limit as $m \to \infty$ and use MCT to get

$$\varliminf \mu_n([-p,p]^n) \geq \mu([-p,p]^n)$$

$$\limsup_{n\to\infty} \int f\, d\mu_n \leq \underbrace{\limsup_{n\to\infty} \int f_p \, d\mu_n}_{= \int f_p \, d\mu \ (f_p \in C_c)} + k\left(1 - \mu([-p,p]^k)\right)$$

$p\to\infty$: $\int f_p\, d\mu \xrightarrow{\text{BCT}} \int f\, d\mu$, and $k(1-\mu([-p,p]^k)) \to 0$ (MCT applied to $1_{[-p,p]^k}$ as $p\to\infty$)

$\triangle$

Let $C_0 = \{ f$ continuous & vanishing at $\infty \}$. Is it true that

$\mu_n \xrightarrow{w} \mu$ iff $\int f\, d\mu_n \to \int f \, d\mu \quad \forall f \in C_0$

Ex: Take $\mu_n = \delta_n$. $\int f\, d\mu_n \to 0 \quad \forall f$.

But $\mu_n \not\xrightarrow{w} 0$. Take $f = 1 \in C_b$. $\int f \, d\mu_n = 1 \neq \int f\, d0 = 0$

Characteristic fcns

μ – prob m're on $\mathbb{R}$ $\rightsquigarrow$ moments $EX^1, EX^2, EX^3, \ldots$

A sbe $a_0, a_1, a_2, \ldots \in \mathbb{R}$ $\rightsquigarrow$ gen. fun $\sum_{i\geq 0}^{\infty} a_i t^i$

exp gen fun $\sum_{i\geq 0}^{\infty} a_i \frac{t^i}{i!}$

Moments $\rightsquigarrow$ Moment gen fcn. – exp gen fcn of moments

$$M_X(t) := \sum_{i\geq 0}^{\infty} (E X^i) \frac{t^i}{i!} \underset{\substack{\uparrow \\ \text{not always}}}{=} E\Big(\sum_{i\geq 0}^{\infty} \frac{X^i t^i}{i!}\Big) \underset{\substack{\uparrow \\ \text{not always exists}}}{=} E(e^{tX}) = \int_{\mathbb{R}} e^{tx} \, d\mu(x)$$

Better to work w/ char. fun – better existence, nice properties

$$\underset{\substack{\uparrow \\ \text{same as Fourier transform}}}{E(e^{itX})} = E(\cos tX) + i E(\sin tX).$$

Defn: The Fourier transform of a prob m're μ on $\mathbb{R}$ is

$$\hat{\mu}(t) := \int_{\mathbb{R}} e^{itx} \mu(dx) \qquad \forall t \in \mathbb{R}$$

If μ – Lebesgue int'ble fcn f, get $\hat{f}(t) = \int_{\mathbb{R}} e^{itx} f(x) \, dx.$

Ex) If $X \sim N(\mu, \sigma^2)$ then $E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

$y = \frac{x-\mu}{\sigma}$ $\qquad \int_{-\infty}^{\infty} e^{ity\sigma + it\mu} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$

$x = y\sigma + \mu$

$$e^{it\mu} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2 + ity\sigma} dy = e^{it\mu - \frac{1}{2}t^2\sigma^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y - it\sigma)^2} dy$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty - it\sigma}^{\infty - it\sigma} e^{-\frac{1}{2}z^2} dz \underset{\substack{\uparrow \\ \text{contour integral}}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz = 1.$$

Characteristic fns

We have seen these before from Le Gall.

1) $X \sim N(\mu, \sigma^2)$ $\quad \phi_X(t) = e^{i\mu t - \sigma^2 t^2/2}$

2) $\phi_X(t)$ determines μ_X $\quad \phi_X(t) = \int e^{itx} \mu_X(dx) = \hat{\mu}_X(t)$

3) $E[f(x-X)] \xrightarrow[\sigma \downarrow 0]{} f(x) \quad \forall f \in C_b(\mathbb{R}) \quad X \sim N(0, \sigma^2)$

$$\overset{\shortparallel}{\int f(x-y)\, g_\sigma(y)\, dy} \longrightarrow f(x) \qquad (\text{Uni}$$

$$\overset{\shortparallel}{f * g_\sigma(x)}$$

(See Fejer thm below. It says $f * g_\sigma \to f$ uniformly on $\mathbb{R}$.)

4) Mollifiers: If $f_\sigma(x) = g_\sigma * \mu(x) = \int g_\sigma(x-y)\mu(dy)$

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

Then $\mu_\sigma(dx) = f_\sigma(x)\,dx$ is the mollified version of μ. It is AC wrt Lebesgue measure.

$$f_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int e^{i\xi x} \underbrace{g_{1/\sigma}(\xi)\, \phi_{-X}(-\xi)}_{\hat{\mu}_\sigma \,=\, \widehat{g * \mu} \,=\, \hat{g}\hat{\mu}} d\xi$$

The density of the mollified measure can be obtained by Fourier inversion

Finally, $\varphi \in C_b(\mathbb{R})$

$$\int \varphi \, \mu_\sigma(dx) = \int \varphi \, f_\sigma(x)\, dx = \int \varphi(x)\, g_\sigma * \mu(x)\, dx$$

$$= \int \varphi(x) \int g_\sigma(x-y)\,\mu(dy)\, dx = \int \varphi * g_\sigma(y)\,\mu(dy)$$

$$\Rightarrow \int \varphi \mu_\sigma(dx) \longrightarrow \int \varphi \, \mu(dy) \quad \text{(from the last equality)}$$

This part of the theorem:

$$\int \varphi \, \mu_\sigma(dx) = \int \varphi(x) f_\sigma(x)\, dx$$

$$= \int \varphi(x) \frac{1}{\sqrt{2\pi\sigma^2}} \int e^{i\xi x} g_{1/\sigma}(\xi)\, \hat{\varphi}_{-x}(-\xi)\, d\xi \, dx$$

$$\boxed{\int \varphi \mu_\sigma(dx) = \frac{1}{\sqrt{2\pi\sigma^2}} \int \hat{\varphi}\, \hat{g}_\sigma \, \overline{\hat{\mu}(\xi)}\, d\xi}$$

The σ^2 cancels with $1/\sigma^2$

Plancharel's Theorem: If $\varphi \in C_c(\mathbb{R})$ and $\hat{\varphi} \in L^1$

$$\int \varphi \, \mu(dx) = \frac{1}{2\pi} \int \hat{\varphi}(\xi)\, \hat{\mu}(\xi)\, d\xi$$

$$\text{"}(\varphi, \mu)\text{"} = (\hat{\varphi}, \hat{\mu})_{L^2(\mathbb{R}, \frac{d\lambda}{2\pi})} \qquad \lambda = \text{Lebesgue meas.}$$

If μ has density f wrt Lebesgue then "(φ, μ)" $= (\varphi, f)_{L^2(\mathbb{R}, d\lambda)}$

So the Fourier transform is (almost) an isometry.
You will see something similar when doing stochastic integrals.

Ex?

Lemma: If μ is a finite m're on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then 1) $\hat{\mu}$ exists,
2) is unif. cts on $\mathbb{R}$ &
3) $\sup_{t\in\mathbb{R}} |\hat{\mu}(t)| = \hat{\mu}(0) = \mu(\mathbb{R})$ & $\hat{\mu}(-t) = \overline{\hat{\mu}(t)}$

4) $\hat{\mu}$ is non-negative definite. i.e. $\forall z_1, \dots, z_n \in \mathbb{C}$, $t_1, \dots, t_n \in \mathbb{R}$
$$\sum_{j=1}^{n}\sum_{k=1}^{n} \hat{\mu}(t_j - t_k)\, z_j \bar{z}_k \geq 0.$$

(In fact Thm: (Bochner) Fourier transform gives a bij b/w normalised, cts, non-neg. defn fns & prob. m'res.)

Pf: WLOG $\mu(\mathbb{R}) = 1$. Let X be a RV w/ distn μ.
1) $\cos tX$ & $\sin tX$ always defined since bdd $\Rightarrow \hat{\mu}(t) = \mathbb{E} e^{itX}$ also
2) $\sup_{|s-t|\leq\delta} |\hat{\mu}(t) - \hat{\mu}(s)| \leq \sup_{|s-t|\leq\delta} \mathbb{E}|e^{itX} - e^{isX}| = \sup_{|s-t|\leq\delta} \mathbb{E}|1 - e^{i(s-t)X}|$

$|1 - e^{it}| = \left|\int_0^t e^{ix}\,dx\right| \leq |t|$ so $\leq \sup_{|s-t|\leq\delta} \underbrace{\mathbb{E}(\delta|X| \wedge 2)}$ $\xrightarrow{\delta\to 0 \text{ by DCT}} 0$
indep of s, t. so get unif. cty.

3) $\hat{\mu}(t) = \sup_{t\in\mathbb{R}} |\hat{\mu}(t)| \leq \int_{\mathbb{R}} 1\, \mu(dx) = 1 = \hat{\mu}(0)$, $\hat{\mu}(-t) = \overline{\hat{\mu}(t)}$

4) $\sum_{j=1}^{n}\sum_{k=1}^{n} \hat{\mu}(t_j - t_k)\, z_j \bar{z}_k = \sum_{j=1}^{n}\sum_{k=1}^{n} \mathbb{E}\left(e^{i(t_j - t_k)X} z_j \bar{z}_k\right)$
$$= \mathbb{E}\left(\sum_{j=1}^{n} e^{it_jX} z_j\right)^2 \geq 0$$
$\triangle$

It is possible to construct a distn from its char fn.

Defn: Suppose $f, g: \mathbb{R} \to \mathbb{R}$ are m'le. Then, when defined, the convolution $f * g$ is the fn
$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy.$$
For $f, g: \mathbb{R}^n \to \mathbb{R}$, similarly define $(f*g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\, dy$.

* Can ignore if you did characteristic fns from Legall

Note: Ch-of-var $z = x-y$ gives $(f*g)(x) = (g*f)(x)$
(i.e. if one side $\exists$, so does the other & equal).

Can use convol. to smoothly approximate fcns.
Let $\phi_\varepsilon(x) = \frac{1}{\varepsilon\sqrt{2\pi}} e^{-\frac{x^2}{2\varepsilon^2}}$.

$\phi_\varepsilon'(x) = -\phi_\varepsilon(x)\left(\frac{x}{\varepsilon}\right)$, etc.

* Ex: Prove $\phi_\varepsilon^{(k)}(x)$ are all uniformly bounded in x.

Thm: (Fejer). If $f \in C_c(\mathbb{R})$, then $f*\phi_\varepsilon$ is ∞'ly diff'ble $\forall \varepsilon>0$
& 1) $(f*\phi_\varepsilon)^{(k)} = f*\phi_\varepsilon^{(k)}$ ($^{(k)}$ - kth derivative) $\forall k \geq 1$.

Moreover 2) $\lim_{\varepsilon\to 0} \sup_{x\in\mathbb{R}} |(f*\phi_\varepsilon)(x) - f(x)| = 0$.

Pf: 1) Use induction. $k=0$ trivial.

$$\frac{(f*\phi_\varepsilon)^{(k)}(x+h) - (f*\phi_\varepsilon)^{(k)}(x)}{h} = \frac{(f*\phi_\varepsilon^{(k)})(x+h) - (f*\phi_\varepsilon^{(k)})(x)}{h} = \int_{-\infty}^{\infty} \underbrace{f(y)}_{\text{compact support}} \underbrace{\frac{\phi_\varepsilon^{(k)}(x+h-y) - \phi_\varepsilon^{k}(x-y)}{h}}_{\text{bdd}} dy$$

BCT gives $\xrightarrow{h\to 0} (f*\phi_\varepsilon^{(k+1)})(x)$

Need Taylor's expansion: if
f is differentiable and $\|f'\|_\infty < K$ then
$|f(x) - f(y)| \leq K|x-y|$

2) Let $Z \sim N(0,1)$. Then εZ has density ϕ_ε so

$$(f * \phi_\varepsilon)(x) = \mathbb{E} f(x - \varepsilon Z)$$

f - unif cts $\Rightarrow$ $\lim\limits_{\varepsilon \to 0} \sup\limits_{x \in \mathbb{R}} |f(x - \varepsilon Z) - f(x)| = 0$ a.s.

f - bdd, so BCT $\quad \mathbb{E}\left(\sup\limits_{x \in \mathbb{R}} |f(x-\varepsilon Z) - f(x)|\right) \geq \sup\limits_{x \in \mathbb{R}} |\mathbb{E}(f(x-\varepsilon Z)) - f(x)|$

$$= \sup_{x \in \mathbb{R}} |f * \phi_\varepsilon(x) - f(x)| \geq 0$$

Take $\lim\limits_{\varepsilon \downarrow 0} \mathbb{E}\left(\sup\limits_{x \in \mathbb{R}} |f(x-\varepsilon Z) - f(x)|\right) = \mathbb{E} \lim\limits_{\varepsilon \to 0} \sup\limits_{x \in \mathbb{R}} |f(x - \varepsilon Z) - f(x)|$

For $\mathbb{R}^n$ can replace ϕ_ε by $\phi_\varepsilon^{(n)}(x) = \frac{1}{\varepsilon (\sqrt{2\pi})^n} e^{-\frac{x_1^2 + \dots + x_n^2}{2}}$

The proof carries through.

Thm: (Plancherel) (Can express f i.t.o. $\hat{f}$).

If μ - finite m're on $\mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$ Lebesgue int'le, then

1) $$\int_{-\infty}^{\infty} (f * \phi_\varepsilon)(x) \, \mu(dx) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\varepsilon^2 t^2/2} \hat{f}(t) \overline{\hat{\mu}(t)} \, dt \quad \forall \varepsilon > 0$$

2) Thus, if $\underbrace{f \in C_b(\mathbb{R})}_{\text{needed to send } \varepsilon \downarrow 0 \text{ on LHS 1)}}$ & $\underbrace{\hat{f} \in L^1(\mathbb{R})}_{\text{needed to send } \varepsilon \downarrow 0 \text{ on RHS 1)}}$, then $\displaystyle\int_{-\infty}^{\infty} f \, d\mu = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t) \overline{\hat{\mu}(t)} \, dt.$

Pf: 7) Fejer + 1) + BCT + $|\hat{\mu}(t)| \le \hat{\mu}(0)$ gives 2).

1)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\varepsilon^2 t^2/2} \hat{f}(t)\overline{\hat{\mu}(t)}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\varepsilon^2 t^2/2} \int_{-\infty}^{\infty} f(x) e^{itx}dx \int_{-\infty}^{\infty} e^{-ity}\mu(dy)\,dt$$

L', BDD

$$\underset{\text{Fubini}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} e^{-\varepsilon^2 t^2/2} e^{it(x-y)}\,dt}_{\frac{\sqrt{2\pi}}{\varepsilon} e^{-\frac{(x-y)^2}{2\varepsilon^2}} = 2\pi\,\phi_\varepsilon(x-y)}\, f(x)dx}_{f * \phi_\varepsilon(y)}\ \mu(dy)$$

This is why important to do normal distribution earlier

$\triangle$

Rmk: "The more derivatives f has, the faster $\hat{f}$ decays at ∞"

If f, f' int'ble, $\hat{f'}(\xi) = i\xi \hat{f}(\xi)$ so $\hat{f}(\xi) = \frac{\hat{f'}(\xi)}{i\xi}$ $\Rightarrow$ decays at least like $\frac{1}{\xi}$ at infty. $f, f', f'' \in L^1 \Rightarrow \hat{f}(\xi)$ decays at least like $\frac{1}{\xi^2} \Rightarrow \in L^1$.

Rmk: There is a similar inversion formula for Fourier transform on $\mathbb{R}^n$.

Cor: (uniqueness thm) If μ, ν are 2 finite m'res on $\mathbb{R}$ then if $\hat{\mu} = \hat{\nu}$, then $\mu = \nu$.

Pf: Let $f \in C_c(\mathbb{R})$. $\int f * \phi_\varepsilon\, d\mu = \int f * \phi_\varepsilon\, d\nu$ (1st part of Plancherel)

$f * \phi_\varepsilon \to f$ unif. $\Rightarrow \int f\, d\mu = \int f\, d\nu$. (Fejer)

Choose $f \in C_c(\mathbb{R})$, $f \downarrow 1_{(a,b]}$.

BCT gives $\int 1_{(a,b]} d\mu = \int 1_{(a,b]} d\nu$

$\mu((a,b]) = \nu((a,b])$

$\{(a,b]\}$ generate $\mathcal{B}(\mathbb{R}) \Rightarrow \mu = \nu$. (MCT)

$\triangle$

★ This is also in Le Gall.

Thm: (Lévy's cvge thm).

Let $\mu, \mu_1, \mu_2, \dots$ be prob measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. ← important

If $\hat{\mu}_n \to \hat{\mu}$ ptwise then $\mu_n \Rightarrow \mu$.

Pf: ETS $\int f \, d\mu_n \to \int f \, d\mu$ $\forall f \in C_c(\mathbb{R})$. (we proved a theorem that said $\mu_n \to \mu$ weakly $\Leftrightarrow$ $\int f d\mu_n \to \int f d\mu$ $\forall f \in C_c(\mathbb{R})$)

$\forall \delta > 0 \; \exists \varepsilon > 0$ s.t. $\sup_{x \in \mathbb{R}} |(f * \phi_\varepsilon)(x) - f(x)| \leq \delta$ (Fejer)

$$\widehat{f * \phi_\varepsilon} = \hat{f} e^{-\varepsilon^2 t^2/2}$$

Take $n \to \infty$ use the fact that $\hat{\mu}_n(t)$ bounded

Plancherel

$$\left| \int f \, d\mu_n - \int f \, d\mu \right| \leq 2\delta + \left| \int (f * \phi_\varepsilon) \, d\mu_n - \int (f * \phi_\varepsilon) \, d\mu \right| = 2\delta + \left| \int_{-\infty}^{\infty} \hat{f}(t) e^{-\frac{\varepsilon^2 t^2}{2}} \frac{\overline{\hat{\mu}_n(t) - \hat{\mu}(t)}}{2\pi} dt \right|$$

$f \in C_c(\mathbb{R}) \Rightarrow \hat{f}$ uniformly bdd by $\int_{\mathbb{R}} |f(x)| \, dx < \infty \Rightarrow$ DCT gives $\limsup_{n \to \infty} \left| \int f \, d\mu_n - \int f \, d\mu \right| \leq 2\delta$

$\triangle$

Rmk: in d-dim'l Fourier inversion implies that for m'res μ_n, μ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $\hat{\mu}_n \to \hat{\mu}$ pointwise implies $\mu_n \Rightarrow \mu$.

It's important to remark that $|\hat{\mu}_n(t)| \leq \hat{\mu}_n(0) = \mu_n(\mathbb{R}) = 1$ be uniformly bounded. Otherwise we cannot send $n \to \infty$ inside the integral.

Remark: Fejér was also important in Lévy's theorem. This is because we cannot directly use

$$\int f \, d\mu_n = \frac{1}{2\pi} \int \hat{f}(t) \, \hat{\mu}_n(t) \, dt$$

since we do not know that $\hat{f}(t) \in L^1$ a priori. The extra $e^{-\frac{\varepsilon^2 t^2}{2}}$ keeps things integrable.

1-D CLT

Thm: $\{X_i\}_{i\ge 1}^{\infty}$ iid, 2 finite moments. $S_n = X_1+\dots+X_n$. $\mathrm{Var}X_1 > 0$

then $\dfrac{S_n - nEX_1}{\sqrt{n}} \Rightarrow N(0, \mathrm{Var}X_1)$

Pf: WLOG Center & scale to get mean 0, var 1.

MTS $\dfrac{S_n}{\sqrt{n}} \Rightarrow N(0,1)$.

ETS Char fn of $\dfrac{S_n}{\sqrt{n}} \to \phi_1(t) = e^{-\frac{t^2}{2}}$.

$$\| $$

$$E(e^{it\frac{S_n}{\sqrt{n}}}) = \prod_{j=1}^{n} E e^{itX_j/\sqrt{n}} = \left(E e^{\frac{itX_1}{\sqrt{n}}}\right)^n$$

Taylor: $e^{ix} = 1 + ix - \frac{1}{2}x^2 + R(x)$

$$E(e^{it\frac{S_n}{\sqrt{n}}}) = \left(1 - \frac{t^2}{2n} + E\left(R\left(\frac{tX_1}{\sqrt{n}}\right)\right)\right)^n.$$

$|R(x)| \le \frac{1}{6}|x|^3 \le |x|^3$. If $|x|>1$, need a diff estimate

$|R(x)| = |e^{ix} - 1 + ix + \frac{1}{2}x^2| \le 1 + 1 + |x| + \frac{1}{2}|x|^2 \le x^2$

So $|R(x)| \le |x|^3 \wedge x^2$.

Skip If doing Lindeberg version as well

$$nE\left(R\left(\frac{tX_1}{\sqrt{n}}\right)\right) \le E\left(\frac{|tX_1|^3}{\sqrt{n}} \wedge |tX_1|^2\right) \xrightarrow{n\to\infty} 0$$

by DCT ($E|X_1|^2 < \infty$).

So $E\left(R\left(\frac{tX_1}{\sqrt{n}}\right)\right) = o\left(\frac{1}{n}\right)$.

So $E(e^{itS_n/\sqrt{n}}) = \left(1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n$

$$\ln\left(E(e^{itS_n/\sqrt{n}})\right) = n\ln\left(1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right) = n\left(-\frac{t^2}{2n} + o\left(\frac{1}{n}\right) + o(\ )\right) \xrightarrow{n\to\infty} -\frac{t^2}{2}$$

so $E(e^{it\frac{S_n}{\sqrt{n}}}) \to e^{-\frac{t^2}{2}}$

Ex:

$$\int_0^1 x e^{itx} dx + \int_1^2 (2-x) e^{itx} dx \qquad \text{(} f \text{ is the tent function)}$$

$$\int x \cos tx = \frac{x \sin tx}{t}\Big|_0^1 - \int_0^1 \frac{\sin tx}{t} dx = \frac{\sin t}{t} + \frac{\cos t - 1}{t^2} = a(t)$$

$$\int x \sin tx = -\frac{x \cos tx}{t}\Big|_0^1 + \int_0^1 \frac{\cos tx}{t} dx = \frac{-\cos t}{t} + \frac{\sin t - 1}{t^2} = b(t)$$

$$\int_1^2 (2-x) e^{itx} dx \qquad 2-x=u \qquad \int_0^1 u\, e^{it(2-u)} du$$

$$= e^{2it} \int_0^1 u e^{-itu} du$$

$$\hat{f}(t) = \big(a(t) + ib(t)\big) + e^{2it}\big(a(t) - ib(t)\big)$$

$$= a(t)\left(1 + e^{2it}\right) + i b(t)\left(1 - e^{2it}\right)$$

Remark: A good homework problem would be one where the function has compact support but

Lindeberg-Feller CLT

<u>Thm:</u> $\forall n$ let $X_{n,1}, \dots, X_{n,n}$ be indep RVs w/ $EX_{nm} = 0$. suppose

1) $\sum_{m=1}^{n} EX^2_{n,m} \to \sigma^2 > 0$

2) $\forall \varepsilon > 0 \ \lim_{n\to\infty} \sum_{m=1}^{n} E(|X_{n,m}|^2 1_{|X_{n,m}|>\varepsilon}) = 0$

Then $S_n = X_{n1} + \cdots + X_{nn} \Rightarrow \sigma N(0,1)$.

<u>Rmk:</u> 1) is saying total variance of const order

2) is saying no summand has large contribution

<u>Remark:</u> It's worth stating, but maybe not worth proving.

So sum of small indep. increments cv to gaussian

<u>pf:</u> let $\varphi_{n,m}(t) = E(e^{itX_{n,m}})$

$$\sigma_{n,m}^2 = E X_{n,m}^2.$$

Then $\varphi_{S_n}(t) = \prod_{m=1}^{n} \varphi_{n,m}(t)$ so ETS

$$\prod_{m=1}^{n} \varphi_{n,m}(t) \longrightarrow e^{-t^2\sigma^2/2}$$

$\uparrow$ is limit of

$$\prod_{m=1}^{n} \underbrace{e^{-t^2\sigma_{n,m}^2/2}}_{\sim 1 - t^2\sigma_{n,m}^2/2}$$

As in CLT

$$\left|\varphi_{n,m}(t) - \left(1 - \frac{t^2\sigma_{n,m}^2}{2}\right)\right| = E(|R(tX_{n,m})|)$$

$$\le E(\min(|tX_{n,m}|^3, 2|tX_{n,m}|^2))$$

$$\le E(|tX_{n,m}|^3 1_{|X_{n,m}|\le\varepsilon}) + E(|tX_{n,m}|^2 1_{|X_{n,m}|>\varepsilon})$$

$$\le \varepsilon t^3 E(|X_{n,m}|^2 1_{|X_{n,m}|\le\varepsilon}) + 2t^2 E(|X_{n,m}|^2 1_{|X_{n,m}|>\varepsilon})$$

$\sum$ over m, $n\to\infty$ get $\sum|\varphi_m - | \le \varepsilon t^3\sigma^2$,

$\forall \varepsilon > 0$ so $\nearrow$ cv to 0.

So $\sum \left| \varphi_{n,m}(t) - \left(1 - \frac{t^2\sigma_{n,m}^2}{2}\right) \right| \to 0.$

$$\left| \prod \varphi_{n,m}(t) - \prod \left(1 - \frac{t^2\sigma_{n,m}^2}{2}\right) \right|$$

$$|\varphi_{n,m}(t)| \le 1, \quad \left| 1 - \frac{t^2\sigma_{n,m}^2}{2} \right| \le 1 \text{ for } n \text{ large}$$

$$\Rightarrow \le \sum \left| \varphi_{n,m} - \left(1 - \frac{t^2\sigma_{n,m}^2}{2}\right) \right| \to 0$$

replace 1 at a time

left to show

$$\prod \left(1 - \frac{t^2\sigma_{n,m}^2}{2}\right) \to e^{-\frac{t^2\sigma^2}{2}}.$$

ie $\sum \ln \left(1 - \frac{t^2\sigma_{n,m}^2}{2}\right) \to e^{-\frac{t^2\sigma^2}{2}}$

Know $\sup_m \frac{t^2\sigma_{n,m}^2}{2} \to 0$

$$\sum_m \frac{t^2\sigma_{n,m}^2}{2} \to \frac{t^2\sigma^2}{2}$$

rest exercise. $\triangleright$

Multi dim'l CLT

Thm Suppose $\{X^i\}_{i=1}^{\infty}$ are iid RVs in $\mathbb{R}^d$ w/ $\mathbb{E}X_i^i = \mu_i$ & $\mathrm{Cov}(X_1^i, X_1^j) = Q_{ij}$. Suppose Q is invertible. Let $S_n = X_1 + \dots + X_n$

Then
$$\lim_{n\to\infty} P\left(\frac{S_n - n\mu}{\sqrt{n}} \in G\right) = \int_G \frac{e^{-\frac{1}{2} y^{tr} Q^{-1} y}}{(2\pi)^{d/2}\sqrt{\det Q}}\, dy$$

i.e. $\dfrac{S_n - n\mu}{\sqrt{n}} \Rightarrow$ multidim'l normal w/ mean 0 & Covar. Q.

Pf: Let $X_n' = X_n - \mu$ so $\mathbb{E}X_n' = 0$.

Let $t \in \mathbb{R}^n$ Then $\mathbb{E}(t \cdot X_n') = 0$ &

$$\mathrm{Var}(t\cdot X_n') = \underbrace{\sum_{i,j=1}^{n} t_i t_j Q_{ij}} \Rightarrow \text{by 1D CLT}$$

$$t \cdot \frac{S_n'}{\sqrt{n}} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} t\cdot X_i' \Rightarrow N(0, \ \cdot\ )$$

$$\Rightarrow \quad \mathbb{E}\left(e^{i t\cdot \frac{S_n'}{\sqrt{n}}}\right) \longrightarrow e^{-\sum_{i,j=1}^{n} t_i t_j Q_{ij}/2}$$

limit is the char fcn of multidim'l normal w/ cov. matrix Q $\triangleright$

Thm: (The Cramér-Wold Device) Let X_n, X be $\mathbb{R}^d$ valued RVs.

$X_n \Rightarrow X$ iff $\forall t \in \mathbb{R}^d$, $t \cdot X_n \Rightarrow t \cdot X$

Pf: $(\Rightarrow)$ $X_n \Rightarrow X$. Let $f \in C_b(\mathbb{R})$. $f(t \cdot \cdot)$ bdd cts fcn on $\mathbb{R}^n$

$\Rightarrow \mathbb{E} f(t \cdot X_n) \to \mathbb{E} f(t \cdot X)$ $\Rightarrow t \cdot X_n \Rightarrow t \cdot X$.

$(\Leftarrow)$ $t \cdot X_n \Rightarrow t \cdot X$ $\forall t \Rightarrow \mathbb{E}(e^{is\, t \cdot X_n}) \to \mathbb{E}(e^{is\, t \cdot X})$ $\forall s, t$

$\Rightarrow \mathbb{E}(e^{i t \cdot X_n}) \to \mathbb{E}(e^{i t \cdot X})$ $\forall t$, i.e. $\hat{\mu}_n \to \hat{\mu}$ $\Rightarrow \mu_n \Rightarrow \mu$

Projective CLT

Another natural situation where Normal distr arises.

$S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ – unit sphere

A set is open iff it comes from an open set of $\mathbb{R}^n$.

Defn: A m're μ on $\mathcal{B}(S^{n-1})$ is a uniform distr on S^{n-1} if

1) $\mu(S^{n-1}) = 1$
2) μ is rotation invariant, i.e. $\forall A \in \mathcal{B}(S^{n-1})$ $\forall$ rotation matrix R ($R^T R = I$), $\mu(A) = \mu(MA)$.

If X has distr μ, say X is uniformly distr. on S^{n-1}.

Thm: If $X^{(n)}$ is uniformly distr. on S^{n-1}, then $\sqrt{n}\, X_1^{(n)} \Rightarrow N(0,1)$.

Rmk: Note the different scaling.

pf: Let $\{Z_i\}_{i \geq 1}^{\infty}$ be iid $N(0,1)$.

Let $Z^{(n)} = (Z_1, \dots, Z_n)$.

$X^{(n)} := \dfrac{Z^{(n)}}{\|Z^{(n)}\|}$ $\forall n \geq 1$.

$E[e^{it \cdot Z^{(n)}}] = E[e^{it_1 Z_1}] E[e^{it_2 Z_2}] \dots$

Indep'ce $\Rightarrow$ the char fn of $Z^{(n)}$ is $f(t) := e^{-\|t\|^2/2}$.

f – is rotation inv: $\forall$ rot. R, $f(Rt) = f(t)$

$\Rightarrow$ Char fn of $Z^{(n)}$ is same as char fn of $RZ^{(n)}$

By invertibility, $Z^{(n)}$ & $RZ^{(n)}$ have same distr $\Rightarrow$

the m're induced by $X^{(n)}$ is rotation inv. $\Rightarrow$ uniform.

$\Rightarrow X^{(n)}$ is uniformly distr.

We have $X_1^{(n)} \|Z^{(n)}\| = Z_1^{(n)} = Z_1$.

By Strong LLN $\dfrac{\|Z^{(n)}\|}{\sqrt{n}} \xrightarrow{a.s.} 1 \Rightarrow X_1^{(n)} \sqrt{n} \xrightarrow{a.s.} Z_1$

$\Rightarrow X_1^{(n)} \sqrt{n} \Rightarrow N(0,1)$.

Done except one gap. If we know $X^{(n)} \sim$ unif on S^{n-1}, how do we know $X^{(n)}$ comes from such a construction as $\dfrac{Z^{(n)}}{\|Z^{(n)}\|}$?

Follows from !'ness of uniform m're.

$\triangle$

Thm: $\exists!$ rotation invariant prob m'ure on S^{n-1}.

Pf: $\exists$'ce already in prev. pf.

Suppose both μ & ν unif. on $S^{(n-1)}$.

(Rmk: $\mu(S^{n-1}) = \nu(S^{n-1})$. rot. inv. $\Rightarrow$ $\mu(\frac{1}{2}S^{n-1}) = \nu(\frac{1}{2}S^{n-1})$
$\frac{1}{4}$ $\frac{1}{4}$
etc.

To show $\mu = \nu$ divide S^{n-1} into "small equal" parts & show they should be equal. Use "ε-balls")

Given $\varepsilon > 0$ & $A \subseteq S^{n-1}$, let $K_A(\varepsilon)$ be largest # disjoint open balls of rad ε that can fit in A. If A is closed, it is compact, so $K_A(\varepsilon)$ is finite.

K_A Kolmogorov ε-entropy / complexity / packing # of A.

μ rot. inv. $\Rightarrow$ $\forall$ closed A

where does that come from?

$$K_A(\varepsilon)\,\mu(B_\varepsilon) \leq \mu(A) \leq (K_A(\varepsilon)+1)\,\mu(B_\varepsilon)$$

where $B_\varepsilon = \{x \in S^{n-1} : d(x, (1,0,\dots,0)) \leq \varepsilon\}$ is a "ball of radius ε".

ditto for ν.

Proof is wrong as written.

$\Rightarrow$ if $\nu(A) > 0$,

$$\frac{K_A(\varepsilon)}{K_A(\varepsilon)+1} \frac{\mu(A)}{\nu(A)} \leq \frac{\mu(B_\varepsilon)}{\nu(B_\varepsilon)} \leq \frac{K_A(\varepsilon)+1}{K_A(\varepsilon)} \frac{\mu(A)}{\nu(A)}$$

So
$$\left| \frac{\mu(A)}{\nu(A)} - \frac{\mu(B_\varepsilon)}{\nu(B_\varepsilon)} \right| \leq \frac{1}{K_A(\varepsilon)} \frac{\mu(A)}{\nu(A)}$$

Set $A = S^{n-1}$ get
$$\left| 1 - \frac{\mu(B_\varepsilon)}{\nu(B_\varepsilon)} \right| \leq \frac{1}{K_{S^{n-1}}(\varepsilon)}$$

Thus
$$\left| \frac{\mu(A)}{\nu(A)} - 1 \right| \leq \frac{1}{K_A(\varepsilon)} \frac{\mu(A)}{\nu(A)} + \frac{1}{K_{S^{n-1}}(\varepsilon)} \xrightarrow{\varepsilon \to 0} 0$$

so
$$\left| \frac{\mu(A)}{\nu(A)} - 1 \right| = 0 \quad \forall \text{ closed } A \text{ s.t. } \nu(A) \neq 0.$$

Similarly, $\mu(A) = \nu(A)$ $\forall$ closed A s.t. $\mu(A) \neq 0$

$\Rightarrow$ $\mu(A) = \nu(A)$ $\forall$ closed A $\Rightarrow$ $\forall A \in \mathcal{B}(S^{n-1})$ $\triangle$

This is a more illuminating proof than the 2019 version.

Liapunov's replacement method

Speed of cv'ce & Different pf of CLT

(not nec. identically distr.)

Thm: For $n \geq 1$. Suppose $X_1, \dots, X_n$ are indep mean zero RVs in $L^3(P)$.

Let $S_n = X_1 + \dots + X_n$ & $s_n^2 := \mathrm{Var}\, S_n$.

If f is any 3 times cts'ly diff'ble fcn w/ $M_f := \sup_z |f'''(z)|$ finite,

Then
$$\left| \mathbb{E} f(S_n) - \mathbb{E} f(N(0, s_n^2)) \right| \leq \frac{2 M_f}{3\sqrt{\pi/2}} \sum_{i=1}^n \|X_i\|_3^3.$$

pf: Idea: keep replacing X's by normals, & keep track of errors.

Let $\sigma_i^2 = \mathrm{Var}(X_i)$.

By Taylor
$$\left| f(S_n) - f(S_{n-1}) - X_n f'(S_{n-1}) - \frac{X_n^2}{2} f''(S_{n-1}) \right| \leq \frac{M_f}{6} |X_n|^3.$$

here use indep'ce

$$\left| \mathbb{E} f(S_n) - \mathbb{E} f(S_{n-1}) - \frac{\sigma_n^2}{2} \mathbb{E} f''(S_{n-1}) \right| \leq \frac{M_f}{6} \|X_n\|_3^3.$$

Let $Z_n \sim N(0, \sigma_n^2)$, indep of $X_1, \dots, X_n$. Replace X_n by Z_n

get
$$\left| \mathbb{E} f(S_{n-1} + Z_n) - \mathbb{E} f(S_{n-1}) - \frac{\sigma_n^2}{2} \mathbb{E} f''(S_{n-1}) \right| \leq \frac{M_f}{6} \|Z_n\|_3^3.$$

so
$$\left| \mathbb{E} f(S_n) - \mathbb{E} f(S_{n-1} + Z_n) \right| \leq \frac{M_f}{6} \Big(\|X_n\|_3^3 + \underbrace{\|Z_n\|_3^3}_{= A \sigma_n^3} \Big)$$

$A \sigma_n^3$, $A = \mathbb{E}|N(0,1)|^3 = \frac{2}{\sqrt{\pi/2}} > 1$

Jensen $\Rightarrow \sigma_n^3 = \left(\mathbb{E}|X_n|^2 \right)^{3/2} \leq \left(\mathbb{E}|X_n|^3 \right) = \|X_n\|_3^3$

So
$$\left| \mathbb{E} f(S_n) - \mathbb{E} f(S_{n-1} + Z_n) \right| \leq \frac{A+1}{6} M_f \|X_n\|_3^3$$

Iterate: replace X_{n-1} by indep Z_{n-1}, etc. get

$$\left| \mathbb{E} f(S_n) - \mathbb{E} f(\underbrace{Z_1 + \dots + Z_n}_{N(0, s_n^2)}) \right| \leq \frac{A+1}{6} M_f \left(\sum_{i=1}^n \|X_n\|_3^3 \right)$$

$$\frac{A+1}{6} < \frac{2}{3\sqrt{\pi/2}}$$

$\triangle$

i.i.d. case immediate

Cor: If $\{X_i\}_{i\in\mathbb{N}}$ iid w/ $\mathbb{E}X_1=0$, $\mathrm{Var}(X_1)=\sigma^2$, & $\mathbb{E}|X_1|^3<\infty$

then $\forall$ g - 3 times ctsly diff'le

$$\left|\mathbb{E}\, g\left(\frac{S_n}{\sqrt{n}}\right) - \mathbb{E}g(N(0,\sigma^2))\right| \le 2\sup_z|g'''(z)| \frac{(\mathbb{E}|X_1|^3)^{1/3}}{3\sqrt{\pi/2}} \frac{1}{\sqrt{n}}.$$

Cor: If $X_1\in L^3(P)$, CLT holds, moreover rate of cv'ce is $\frac{1}{\sqrt{n}}$.

Pf: Set $g(x)=e^{itx}$ $\triangle$

In fact Liapunov gives CLT even for $X_1\in L^2(P)$

(Lindeberg's pf of CLT)

Use truncation method

WLOG $\sigma=1$. let $\varepsilon>0$. set $X_i' := X_i \mathbb{1}_{|X_i|\le\varepsilon\sqrt{n}}$, $S_n'=\sum_{i=1}^n X_i'$, $M_n=\mathbb{E}S_n'$,

$s_n^2=\mathrm{Var}\, S_n'$.

Let g be 3 times diff'le w/ cts bdd derivs.

By Liapunov

only depends on $\max_z|g'''(z)|$

$$(*) = \left|\mathbb{E}\, g\left(\frac{S_n'-M_n}{\sqrt{n}}\right) - \mathbb{E}\, g\left(N\left(0,\frac{s_n^2}{n}\right)\right)\right| \le \frac{C_g}{\sqrt{n}} \mathbb{E}(|X_1'-\mathbb{E}X_1'|^3)$$

$$\le \frac{8C_g}{\sqrt{n}}\mathbb{E}|X_1'|^3.$$

$$|X_1'|\le\varepsilon\sqrt{n} \Rightarrow \mathbb{E}|X_1'|^3 \le \varepsilon\sqrt{n}\,\mathbb{E}|X_1'|^2 \le \varepsilon\sqrt{n}\,\mathbb{E}|X_1|^2 = \varepsilon\sqrt{n}.$$

So $(*) \le 8C_g\varepsilon.$

Use Taylor to Replace S_n' by S_n:

$$\left|\mathbb{E}g\left(\frac{S_n'-M_n}{\sqrt{n}}\right) - \mathbb{E}g\left(\frac{S_n}{\sqrt{n}}\right)\right| \le \sup_z|g'(z)|\; \mathbb{E}\left|\frac{S_n'-S_n-M_n}{\sqrt{n}}\right| \le \sup_z|g'(z)|\sqrt{\mathrm{Var}\left(\frac{S_n'-S_n-M_n}{\sqrt{n}}\right)}$$

$$S_n-S_n' = \sum_{i=1}^n X_i\mathbb{1}_{|X_i|>\varepsilon\sqrt{n}} \Rightarrow \mathrm{Var}\left(\frac{S_n'-S_n-M_n}{\sqrt{n}}\right) = \frac{1}{n}\cdot n\cdot \mathrm{Var}(X_1\mathbb{1}_{|X_1|>\varepsilon\sqrt{n}})$$

$$= \mathbb{E}|X_1|^2\mathbb{1}_{|X_1|>\varepsilon\sqrt{n}}$$

So $\left|\mathbb{E}g\left(\frac{S_n}{\sqrt{n}}\right) - \mathbb{E}g\left(N\left(0,\frac{s_n^2}{n}\right)\right)\right| \le A\varepsilon +$

$$\frac{s_n^2}{n} = \frac{\mathrm{Var}(S_n')}{n} = \mathrm{Var}(X_1\mathbb{1}_{|X_1|\le\varepsilon\sqrt{n}}) \xrightarrow[n\to\infty]{DCT} \mathrm{Var}(X_1)=1.$$

$$\mathbb{E}\, g\left(N\left(0,\frac{s_n^2}{n}\right)\right) = \mathbb{E}\, g\left(N(0,1)\frac{s_n}{\sqrt{n}}\right) \xrightarrow{n\to\infty} \mathbb{E}\, g(N(0,1))$$

So $\lim_{n\to\infty}\left|\mathbb{E}\, g\left(\frac{S_n}{\sqrt{n}}\right) - \mathbb{E}g(N(0,1))\right| \le A\varepsilon$ $\forall\varepsilon>0 \Rightarrow =0$ $\triangle$

Bochner's Theorem

It's worth pointing out the use of the mollification in Plancharel's theorem: it makes the Fourier transform integrable.

$$\int f * \phi_\varepsilon \, \mu(dx) = \int e^{-\varepsilon^2 t^2/2} \hat{f}(t) \overline{\hat{\mu}(t)} dt \quad -\star 1$$

If $\hat{f} \in L^1$ then

$$\int f \, \mu(dx) = \int \hat{f}(t) \overline{\hat{\mu}(t)} dt$$

This involves taking a limit as $\varepsilon \downarrow 0$ in $(\star 1)$.

Pf: $\phi_n(t) \longrightarrow \phi(t)$ as $n \longrightarrow \infty$

+def = positive definite (nonnegative def. really)

1) If $\phi(t)$ is +def, so is $\phi(t) e^{ita}$ for any real a.

$$\sum \phi(t_i - t_j) e^{i(t_i - t_j)a} z_i \bar{z}_j$$ same proof by absorbing $e^{it_i a}$ in z_i

2) If $\{\phi_j\}_{j=1}^{n}$ +ve def then so is $\sum_{j=1}^{n} \phi_j(t) w_j$ for any nonnegative w_j

3) If ϕ is +ve def

a) $\phi(0) \geq 0$ Choose $t_1 = 0$ $z \phi(0) \bar{z} = \phi(0) |z|^2 \geq 0$

b) $\phi(-t) = \overline{\phi(t)}$ Choose $t_1 = t$, $t_2 = 0$

c) $|\phi(t)| \leq \phi(0)$

$$\begin{bmatrix} \phi(0) & \phi(t) \\ \phi(-t) & \phi(0) \end{bmatrix} \geq 0$$ (Leave as exercise)

$$\Leftrightarrow \quad (\lambda - \phi(0))^2 - |\phi(t)|^2 = 0 \Rightarrow \quad \lambda = \phi(0) \pm |\phi(t)|$$

If both eigenvalues are to be positive, then

$$\phi(0) \geq |\phi(t)| \qquad \forall t$$

Or directly $\det \geq 0 \Rightarrow \quad \phi(0)^2 - |\phi(t)|^2 \geq 0$.

4) $\forall s, t \qquad |\phi(t) - \phi(s)|^2 \leq 4\phi(0)\,|\phi(0) - \phi(t-s)|$

$$t_1 = t \qquad t_2 = s \qquad t_3 = 0$$

$$\begin{array}{c} t \\ s \\ 0 \end{array} \begin{bmatrix} \phi(0) & \phi(t-s) & \phi(t) \\ \overline{\phi(t-s)} & \phi(0) & \phi(s) \\ \overline{\phi(t)} & \overline{\phi(s)} & \phi(0) \end{bmatrix}$$

$$\det = \phi(0)^3 - \phi(0)|\phi(s)|^2 - \phi(t-s)\left(\phi(0)\overline{\phi(t-s)} - \phi(s)\overline{\phi(t)}\right)$$
$$+ \phi(t)\left(\overline{\phi(t-s)}\,\overline{\phi(s)} - \phi(0)\overline{\phi(t)}\right)$$

$$= \phi(0)^3 - \phi(0)|\phi(s)|^2 - |\phi(t-s)|^2\phi(0) + \phi(t-s)\phi(s)\overline{\phi(t)}$$
$$+ \phi(t)\overline{\phi(s)}\,\overline{\phi(t-s)} - \phi(0)|\phi(t)|^2 \qquad \geq 0$$

$$|\phi(t) - \phi(s)|^2 = |\phi(t)|^2 + |\phi(s)|^2 - \phi(t)\overline{\phi(s)} - \phi(s)\overline{\phi(t)}$$

$$\Rightarrow \quad \phi(0)^3 - |\phi(t-s)|^2\phi(0) - \phi(t)\overline{\phi(s)}\,\phi(0) - \phi(s)\overline{\phi(t)}\,\phi(0)$$
$$+ \phi(t-s)\phi(s)\overline{\phi}(t) + \phi(t)\overline{\phi(s)}\,\overline{\phi(t-s)}$$

$$\geq \phi(0)\,|\phi(t)-\phi(s)|^2$$

$$\phi(0)^3 - |\phi(t-s)|^2\phi(0) + \phi(t)\overline{\phi(s)}\left(\overline{\phi(t-s)}-\phi(0)\right) + \phi(s)\overline{\phi(t)}\left(\phi(t-s)-\phi(0)\right)$$

$$\geq \phi(0)\,|\phi(t)-\phi(s)|^2$$

Focus on this part

$$\phi(t)\overline{\phi(s)}\left(\overline{\phi(t-s)}-\phi(0)\right) + \phi(s)\overline{\phi(t)}\left(\phi(t-s)-\phi(0)\right)$$

$$= +2\,\mathrm{Re}\left(\phi(t)\overline{\phi(s)}\left(\overline{\phi(t-s)}-\phi(0)\right)\right)$$

$$\leq 2|\phi(t)||\phi(s)|\,|\phi(t-s)-\phi(0)|$$

$$\Rightarrow \phi(0)^{\cancel{3}\,2} - |\phi(t-s)|^2\cancel{\phi(0)} + 2\phi(0)^{\cancel{2}}|\phi(t-s)-\phi(0)|$$

$$\geq \cancel{\phi(0)}\,|\phi(t)-\phi(s)|^2$$

$$\phi(0)^2 - |\phi(t-s)|^2 + 2\phi(0)|\phi(0)-\phi(t-s)|$$

$$|\phi(0)|^2 - |\phi(t-s)|^2 = \left(|\phi(0)| + |\phi(t-s)|\right)\left(\phi(0)-\phi(t-s)\right)$$

$$\leq 2\phi(0)\,|\phi(0)-\phi(t-s)|$$

$$\Rightarrow |\phi(t)-\phi(s)|^2 \leq 4\phi(0)\,|\phi(0)-\phi(t-s)|$$

$\Rightarrow$ ϕ is UNIFORMLY CONTINUOUS EVERYWHERE.

<u>Step 2</u> If ϕ is a positive definite function continuous on $\mathbb{R}$ and $\underbrace{\text{absolutely integrable}}_{\hat{\mu}\text{ integrability in Fejer}}$ with $\phi(0)=1$, then

(★2) $$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\,\phi(t)\,dt \geq 0 \quad \text{(pdf)}$$

$$\int_{\mathbb{R}} f(x)\,dx = 1$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt \quad \text{is a CDF.}$$

with characteristic function ϕ !

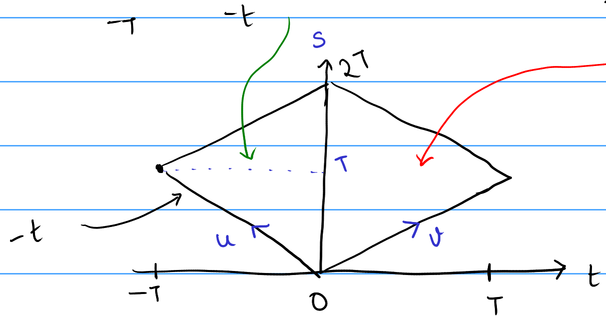
If ϕ is integrable f is bounded as defined by ★2 and its continuous (using DOM)

To see $f \geq 0$

$$f(x) = \underset{T\to\infty}{\overset{\text{DOM}}{\lim}} \frac{1}{2\pi}\int_{-T}^{T}\left(1 - \frac{|t|}{T}\right) e^{-itx}\,\phi(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{0} \frac{T+t}{T} e^{-itx}\,\phi(t)\,dt + \int_{T}^{0} \frac{T-t}{T} e^{-itx}\,\phi(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{0} \frac{1}{2T}\int_{-t}^{2T+t} e^{-itx}\,\phi(t)\,ds\,dt + \frac{1}{2T}\int_{T}^{0}\int_{t}^{2T-t} e^{-itx}\,\phi(t)\,ds\,dt$$



$u = (s-t)/2$ $\quad s = u+v$

$v = (s+t)/2$ $\quad t = u-v$

$$\left|\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right| = 2$$

$$= \lim_{T\to\infty} \frac{1}{2\pi}\frac{1}{2T}\int_0^T\int_0^T e^{-i(v-u)x}\,\phi(v-u)\,2\,du\,dv$$

Jacobian

$$= \lim_{T\to\infty} \frac{1}{2\pi}\underbrace{\int_0^T\int_0^T e^{-ivx} e^{iux}\,\phi(v-u)\,du\,dv}_{\text{using a Riemann sum approximation and using positive definiteness.}} \geq 0$$

To show

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

Set $t=0$ to get $\int f(x)\,dx = 1$

$$f_\sigma(x) = f(x)\,e^{-\frac{\sigma^2x^2}{2}}$$

$$\int_{-\infty}^{\infty} e^{itx} f_\sigma(x)\,dx = \int_{-\infty}^{\infty} e^{itx} f(x)\,e^{-\frac{\sigma^2x^2}{2}}\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{itx}\,\phi(s)\,e^{-isx}\,e^{-\frac{\sigma^2x^2}{2}}\,ds\,dx$$

Fubini (uses integrability of $\phi(s)\,e^{-\frac{\sigma^2x^2}{2}}$

This helps this be integrable)

In Plancharel, our version of

$$f(x) = \widehat{\phi(-x)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(s)\,\frac{1}{\sqrt{2\pi\sigma}}\,e^{-\frac{(t-s)^2}{2\sigma^2}}\,ds$$

$$\Rightarrow \quad \int e^{itx} f_\sigma(x) = \int_{-\infty}^{\infty} \phi(s) \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(t-s)^2}{2\sigma^2}} ds \qquad -(\star 3)$$

Set $t = 0$ to get

$$\int f_\sigma(x) = \int_{-\infty}^{\infty} \phi(s) \frac{1}{\sqrt{2\pi\sigma}} e^{-s^2/2\sigma^2} ds$$

$$\leq \int \frac{1}{\sqrt{2\pi\sigma}} e^{-s^2/2\sigma^2} ds = 1 \quad \text{using } |\phi(s)| \leq 1$$

$$\underline{\lim}_{\sigma \downarrow 0} \int f_\sigma(x) \geq \int f(x)\, dx \qquad \text{From Fatou}$$

$\Rightarrow \quad \int f(x)\, dx \leq 1$ But $f_\sigma(x) \leq f(x)$ and thus by DOM in $(\star 3)$ Take $\sigma \downarrow 0$ to get

$$\int e^{itx} f(x)\, dx = \lim_{\sigma \to 0} E\,\phi(s + \sigma Z)$$

(By Fejér's theorem and integrability of ϕ.)

Step 3: Thus far we worked with

1) ϕ +ve def
2) ϕ integrable

we need to remove the integrability assumption.

By previous $\phi(t)e^{ity}$ is also positive definite.

$$\int \phi(t)e^{ity} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{y^2}{2\sigma^2}} dy = \phi_\sigma(t) \quad \text{is also +ve def}$$

$$\phi(t)\, e^{-\frac{\sigma^2 t^2}{2}} = \phi_\sigma(t) \quad \text{which is integrable.}$$

By previous $\phi_\sigma(t)$ is a characteristic function.

$\lim_{\sigma \to \infty} \phi_\sigma(t) \longrightarrow \phi(t)$ that is positive definite and continuous.
st $\phi(0) = 0$.

There is an improvement of levy continuity that shows that ϕ must be a characteristic fn of a measure as well.
(It's in Varadhan)

<u>Remark</u>: Suppose the characteristic function is integrable, show that the measure you obtain must be absolutely continuous wrt Lebesgue measure. (does the previous prove this?)